AN ANALYSIS OF EQUITY IN INSURANCE. THE
MATHEMATICAL APPROACH OF RISK OF RUIN FOR
INSURERS

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Abstract: The goal of the present paper is a short analysis for the insurers' foreign equity in Romania and the development of a mathematical approach for the chronological evolution of the study regarding the insurers’ equity from the point of view of assessing the insolvency probabilities and the risk provision so the estimating insolvency risk will not over overcome an accepted value.

Key words: Insurer, broker, adjusting coefficient, overcharging factor, probability of ruin, compound Poisson repartition.

Introduction
Since 1989, Romania faced a development of private property and a concentration of indigene capital, facts that determined continuous growth of the insurance market. In a first phase, the permissible legal framework (Law no. 47/1991) and the lack of the Romanian capital but also the attractiveness of the Romanian market were the main factors for the market penetration of the foreign capital on the Romanian insurance market. The same phenomenon took place in the insurance mediation sector.

The number of insurers reached a normal evolution, with a spectacular growth at the beginning of the insurance market, followed by a decrease according to the concentration and centralization of the capital, as shown below: (P is the percent of foreign capital to equity of the insurers).

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of insurers</th>
<th>P %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997</td>
<td>47</td>
<td>-</td>
</tr>
<tr>
<td>1998</td>
<td>64</td>
<td>-</td>
</tr>
<tr>
<td>1999</td>
<td>72</td>
<td>62,11</td>
</tr>
<tr>
<td>2000</td>
<td>73</td>
<td>75,80</td>
</tr>
<tr>
<td>2001</td>
<td>47</td>
<td>61,70</td>
</tr>
<tr>
<td>2002</td>
<td>48</td>
<td>50,20</td>
</tr>
<tr>
<td>2003</td>
<td>44</td>
<td>59,21</td>
</tr>
<tr>
<td>2004</td>
<td>45</td>
<td>48,00</td>
</tr>
<tr>
<td>2005</td>
<td>43</td>
<td>50,00</td>
</tr>
<tr>
<td>2006</td>
<td>41</td>
<td>53,10</td>
</tr>
</tbody>
</table>

Source: - OSAAR Reports, Bucharest, years 1997-1999;
- ISC Reports, Bucharest, years 2000-2006;

As it can be seen, a significant share of equity is represented by the foreign capital, mainly Austrian or German. Since year 2007, it is evident a foreign capital domination due to the fact that the first 6 insurers have a market share of more than 70%. Since year 2000, there are changes in the sales management for the insurance products, so the insurance agent is replaced by the insurance broker. The law 32/2000 sets a clearer view on mediation sector, especially for brokers. This is supposed to be a legal person, to prove
cash equity, to have valid third party liability insurance, and the only subject of activity – the insurance
mediation, to have qualified human resources.

The evolution for the number of brokers is shown below:

<table>
<thead>
<tr>
<th>Year</th>
<th>1999</th>
<th>2000</th>
<th>2001</th>
<th>2002</th>
<th>2003</th>
<th>2004</th>
<th>2005</th>
<th>2006</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of brokers</td>
<td>749</td>
<td>817</td>
<td>198</td>
<td>150</td>
<td>204</td>
<td>266</td>
<td>317</td>
<td>344</td>
</tr>
</tbody>
</table>


It is important to assess the risk that an insurance company faces bankruptcy. In order to explain this, we
shall provide a mathematical model regarding the insurer’s capital and we shall determine the ruin
probability, this means the probability that the insurer has no more resources to pay the indemnities
(obviously this isn’t a de facto ruin, but an attention for the financial management).

2. The modeling of risk of ruin

In order to study the variation of insurer’s equity, we shall note with $U(t)$ the insurer’s equity at moment $t$,
with $u$ the initial capital, with $S(t)$ the total claim demand (more precisely the amount of total claim
demand) occurred till the moment $t$, with $m$ the number of insurance policies, with $N_k(t)$ the number of
individual claim demands till the moment $t$ for a given policy type $k$, with $Y_i^k$ the individual demand
number $i$ for the insurance type $k$ and with $S_k(t)$ the claim demand occurred till the moment $t$ for a
given policy type $k$.

We have:

$$ S(t) = \sum_{k=1}^{m} S_k(t) \quad \text{and} \quad S_k(t) = \sum_{i=1}^{N_k(t)} Y_i^k. $$

Additionally, we presume the following hypothesis to be fulfilled:

1. there are no other expenses than the paid indemnities and no other incomes than the cashed
   premiums.
2. the unit income is $c$.
3. the individual claim demands for every policy type are independent identically distributed
   random variables. The individual claim demands for every insurance type are independent
   and identically distributed variables
4. the stochastic processes $(N_k(t))_t$, $k = 1, m$, for the number of demands are Poisson
   processes of parameter $\lambda_k$. It results that process $(S(t))_t$ is a compound Poisson process
   composed of parameter $\lambda = \sum_{k=1}^{m} \lambda_k$, so we have $S(t) = \sum_{i=1}^{N(t)} X_i$, where random variables
   $X_i$ are independent and identically distributed, having the moment of order $k$ noted with $m_k$,
   $m_k = M(X^k)$. The variables $X_i$ equalize mathematically the individual real claim
   demands $Y_i^k$, but there are not identically distributed. In these hypothesis, we have
   $M(S(t)) = \lambda \cdot t \cdot m_1$ and $D^2(S(t)) = \lambda \cdot t \cdot m_2$.
5. The insurance premiums are determined with respect to the medium value principle, with an
   overcharging relative confidence $\theta$:

$$ k \cdot c = (1 + \theta) \cdot m_1 \cdot M(N(1)). $$

211
We have: \( U(t) = u + c \cdot t - S(t) \) \hspace{1cm} (1).

We shall name ruin the situation this capital is negative and we shall consider the moment of ruin at the moment this happen for the first time. So we are dealing with the problem of ruin for a given, undetermined period of time, called finite horizon. So:

\[
T = \inf \{ t > 0 \mid U(t) < 0 \}
\]

is the moment of ruin.

In this context, the ruin probability (noted with \( \Psi \)) is the probability that the moment of ruin is finite. From the point of view of initial capital \( u \), the overcharging factor \( \theta \), the claim files rate \( \lambda \) and the medium individual claim demand \( m_1 \), the function is: \( \Psi(u, \theta, \lambda, m_1) = P(T < \infty) \).

The equation (in \( r \)) strictly positive solution: \( \lambda + c \cdot r = \lambda \cdot M \left( e^{rX} \right) \) is called adjustment coefficient. This coefficient (when does exist) shall be noted as \( R \).

Considering an existing adjustment coefficient \( R \), we have:

\[
\Psi = \frac{e^{-Ru}}{M \left( e^{-Ru(T)} \mid T < \infty \right)}.
\]

If the adjustment coefficient \( R \) does exist, then \( \Psi(u) \leq e^{-Ru} \) (this is known as the Cramer inequality).

This is determined by the fact that for a finite \( T \) (\( T < \infty \)), we have \( U(T) < 0 \), so \( M \left( e^{-Ru(T)} \mid T < \infty \right) > 1 \).

Generally speaking, because the adjustment coefficient is difficult to compute, we seek for a convenient interval, whose margins are used in the Cramer inequality. Because \( M \left( e^{RX} \right) > 1 + Rm_1 + \frac{R^2}{2}m_2 \) we obtain: \( R < \frac{2 \cdot \theta \cdot m_1}{m_2} \). Considering the function \( \omega(r) = \lambda \cdot (m_2(r) - 1) - c \cdot r \) which is strictly convex and due to the fact that net profit condition is fulfilled \((\omega'(0) = \lambda \cdot m_1 - c < 0)\), the results are \( \omega(R) = \omega(0) + \int_0^R \omega'(s) \cdot ds > \lambda \cdot m_1 \cdot R - c \cdot R + \frac{R^2}{\lambda \cdot m_2} \) and \( R < \frac{2 \cdot c - 2 \cdot \lambda \cdot m_1}{\lambda \cdot m_2} \).

If the individual demands have an exponential repartition of parameter \( \alpha \) (so \( m_1 = \frac{1}{\alpha} \)), because the medium income per unit time is \( c = \left(1 + \theta \right) \cdot \frac{\lambda}{\alpha} \), we find for the ruin probability the expression:

\[
\Psi(u, \theta, \lambda, \alpha) = \frac{\lambda}{\alpha \cdot c} \cdot e^{-\frac{\alpha \cdot \lambda}{c}u} = \frac{1}{1+\theta} \cdot e^{-\frac{\alpha \cdot u}{1+\theta}} \hspace{1cm} (2).
\]

Considering the initial capital as a \( \beta \) number of medium individual claim demands, we obtain:

\[
\Psi(\beta, \theta) = \frac{1}{1+\theta} \cdot e^{-\frac{\beta \cdot \theta}{1+\theta}} \hspace{1cm} (3).
\]

We could improve this model by setting the problem to compute a risk provision \( V \), in order to define the probability that paid indemnities should not overcome the incomes plus this provision, under an accepted value \( p \). If the medium number of claim demands (in our case \( n = \lambda \cdot t \)) is big enough, we can use the central limit theorem (TLC) and approximate \( S(t) \) with a normal variable of mean \( \lambda \cdot t \cdot m_1 \) and dispersion \( \lambda \cdot t \cdot m_2 \). So we want to determine \( V \) in order that \( P(S(t) \geq u + c \cdot t + V) \leq p \). Using TLC
we obtain $V \geq z_{1-p} \cdot \sqrt{\lambda \cdot t \cdot m_2} - u - \theta \cdot \lambda \cdot t \cdot m_1$, where $z_{1-p}$ is the quartile of order $1 - p$ of the normal repartition. If we choose the minimum risk reserve in order not to block higher financial resources, we have:

$$V = z_{1-p} \cdot \sqrt{\lambda \cdot t \cdot m_2} - u - \theta \cdot \lambda \cdot t \cdot m_1 \quad (4)$$

In the hypothesis that individual claim demand follows an exponential repartition of parameter $\alpha$, so $m_1 = \frac{1}{\alpha}$, $m_2 = \frac{1}{\alpha^2}$, we have:

$$V = z_{1-p} \cdot \sqrt{\frac{\lambda \cdot t}{\alpha}} - u - \theta \cdot \frac{\lambda \cdot t}{\alpha} \quad (5)$$

Choosing an initial equity to represent a number of individual medium claim demands ($\beta$), we obtain:

$$V = \frac{z_{1-p} \cdot \sqrt{\alpha \cdot t} - \beta - \theta \cdot \lambda \cdot t}{\alpha} \quad (6)$$

### 3. Numerical results

From relation (3) we have determined the values for ruin probabilities for a few values of premiums overcharging factor $\theta$ and parameter $\beta$, that provide the number of medium individual demands to cover the initial equity.

The results are listed in the table below:

<table>
<thead>
<tr>
<th>$\theta$ \ $\beta$</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>50</th>
<th>60</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.5770</td>
<td>0.3663</td>
<td>0.1476</td>
<td>0.05945</td>
<td>0.00965</td>
<td>0.00428</td>
<td>0.0001</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3622</td>
<td>0.1574</td>
<td>0.0297</td>
<td>0.0056</td>
<td>0.0002</td>
<td>4.5 $\cdot$ 10$^{-5}$</td>
<td>4.8 $\cdot$ 10$^{-8}$</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1712</td>
<td>0.0410</td>
<td>0.0024</td>
<td>0.00014</td>
<td>4.5 $\cdot$ 10$^{-7}$</td>
<td>3.6 $\cdot$ 10$^{-8}$</td>
<td>2.8 $\cdot$ 10$^{-13}$</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0958</td>
<td>0.0147</td>
<td>0.0003</td>
<td>8.1 $\cdot$ 10$^{-6}$</td>
<td>4.5 $\cdot$ 10$^{-9}$</td>
<td>1.7 $\cdot$ 10$^{-10}$</td>
<td>3.2 $\cdot$ 10$^{-17}$</td>
</tr>
</tbody>
</table>

Obviously, the higher the initial equity and the bigger the overcharging premiums are, the smaller the ruin probability is. The diminution of risk of ruin realizes exponentially compared to the growth of those two factors. The increasing overcharging factor is limited by the necessity to maintain an adequate premium level for the competitiveness of the insurance company on market. We also emphasize this model is correlating to strong the incomes and the indemnities, the medium income per unit time being proportional to the medium number of individual claim demands multiplied by the value of medium individual claim demand.

This is due to the principle of medium value for the premiums computation. We consider the use of medium annual claim index instead of claim demand in order to obtain a more realistic model.

Using relation (6) we obtained the following results:

- for $p = 0.005$, $\alpha = 0.001$, $\lambda = 1$, $\beta = 2$, values of $t$ and $\theta$, the risk reserve $V$ (u.m.) is provided in the following table:

<table>
<thead>
<tr>
<th>$t$ \ $\theta$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>480</td>
<td>380</td>
<td>80</td>
</tr>
<tr>
<td>5</td>
<td>3269</td>
<td>2769</td>
<td>1269</td>
</tr>
</tbody>
</table>
for \( p = 0.001, \alpha = 0.01, \lambda = 5, \theta = 0.1 \), values of \( t \) and \( \beta \), the risk reserve \( V \) (u.m.) is provided in the following tabel:

<table>
<thead>
<tr>
<th>( t \backslash \beta )</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1590</td>
<td>1090</td>
<td>90</td>
</tr>
<tr>
<td>400</td>
<td>1680</td>
<td>1180</td>
<td>180</td>
</tr>
</tbody>
</table>

We underline that the negative values for the relations (4)-(6) mean a risk reserve equal to zero. We also ascertain the risk reserve to be proportional to the claim medium individual demand, decreasing when the initial equity covers many claim individual demands and decreasing when the requested margin for the insolvency risk decreases.

References